Computers Math. Applic. Vol. 25, No. 3, pp. 3–11, 1993
Printed in Great Britain.

0097-4943/93 $5.00 + 0.00

FIRST-ORDER ASYMPTOTIC EXPANSION METHOD FOR SINGULARLY PERTURBED SYSTEMS OF SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS

JANUSZ R. MIKA AND JAN M. KOZAKIEWICZ[1]
University of Natal, Department of Mathematics and Applied Mathematics
King George V. Avenue, Durban 4001, South Africa

(Received April 1991)

Abstract—We apply the new algorithm developed by Mika and Palczewski for singularly perturbed systems of ordinary differential equations to systems of second order equations closely related to a discretised version of the telegraph equation.

1. INTRODUCTION

The existing methods of solving initial value problems for ordinary differential equations are so well developed that any arbitrarily large stiff system of equations can be solved numerically with an adequate accuracy if a sufficiently large computing system is available. Such an opinion has been voiced, for instance, at the Conference on Computational Ordinary Differential Equations, London, 3–7 July 1989. On the other hand, with an increasing popularity of personal computers there seems to be still the demand for methods capable of solving fairly large stiff systems of ordinary differential equations without resorting to large mainframe computers. One such method is the now classic algorithm of the asymptotic expansion for initial value problems which has been developed mainly due to Vasil'eva [1] and O'Malley [2] (see also the monograph by Smith [3]). Recently, Mika and Palczewski have proposed, for autonomous systems, a new algorithm which gives the first-order approximation at roughly the same computing effort as the standard approximation of the zero order, usually referred to as the steady-state approximation (see [4–7]).

In this paper, we apply the new algorithm to a particular system of second order differential equations which is closely related to the discretised version of the telegraph equation.

2. FORMULATION OF THE PROBLEM

We consider the initial value problem

$$\epsilon\frac{d^2x}{dt^2} + A\frac{dx}{dt} + f(x) = 0, \qquad x(0) = \alpha, \quad \frac{dx}{dt}(0) = \beta, \tag{2.1}$$

where

$$t \in [0, t_1],\ t_1 > 0, \quad x : [0, t_1] \to \mathbb{R}^n,\ n \geq 1, \quad f : \mathbb{R}^n \to \mathbb{R}^n, \quad \alpha, \beta \in \mathbb{R}^n,$$

and ϵ is a small positive parameter and A is a matrix whose eigenvalues have all positive real parts. We note that A is invertible.

The second-order system may be converted to a first-order system simply by treating $\frac{dx}{dt}$ as a new unknown function, so that (2.1) is written as

$$\epsilon\frac{dz}{dt} = -Az - f(x), \qquad \frac{dx}{dt} = z, \qquad x(0) = \alpha,\ x(0) = \beta. \tag{2.2}$$

[1] On leave of absence from Institute of Atomic Energy, Swierk 05400, Otwock, Poland.

Alternatively, we may define a new function $v : [0, t_0] \to \mathbb{R}^n$ by

$$v(t) = Ax(t) + \epsilon \frac{dx}{dt}(t),$$

so that (2.1) takes the form

$$\epsilon \frac{dx}{dt} = -Ax + v, \quad \frac{dv}{dt} = -f(x), \quad x(0) = \alpha, \quad v(0) = A\alpha + \epsilon\beta. \tag{2.3}$$

We will apply the asymptotic expansion algorithm to both systems (2.2) and (2.3).

3. ALGORITHM

The new algorithm of the asymptotic expansion proposed by Mika and Palczewski is described in detail in [6] (see also [5]). Here we give only those features of the algorithm which are pertinent to the present problem. In particular, we will truncate the expansions at first-order terms in order to derive the first-order version of the steady-state approximation.

First consider the system (2.2). Each of the functions x and z will be represented as a sum of a function depending on t (bulk solution) and of a function dependent on $\tau = t/\epsilon$ (initial layer solution) and then both will be expanded in powers of ϵ. In the first-order approximation, we will have

$$z(t) = \overline{z}^{(1)}(t) + \widetilde{z}^{(1)}(\tau) + 0(\epsilon^2), \qquad x(t) = w(t) + \widetilde{x}^{(1)}(\tau) + 0(\epsilon^2), \tag{3.1}$$

where

$$\overline{z}^{(1)}(t) = \overline{z}_0(t) + \epsilon \overline{z}_1(t), \quad \widetilde{z}^{(1)}(\tau) = \widetilde{z}_0(\tau) + \epsilon \widetilde{z}_1(\tau), \quad \widetilde{x}^{(1)}(\tau) = \widetilde{z}_0(\tau) + \epsilon \widetilde{z}_1(\tau). \tag{3.2}$$

The characteristic feature of the algorithm proposed by Mika and Palczewski is that the bulk solution w for the slow variable x remains unexpanded.

Since the original system is autonomous, the bulk solution for the fast variable will depend on time through its functional dependence on w. Introducing two functions $\varphi_0 : \mathbb{R}^n \to \mathbb{R}^n$ and $\varphi_1 : \mathbb{R}^n \to \mathbb{R}^n$ we can write

$$\overline{z}_0(t) = \varphi_0(w(t)), \quad \overline{z}_1(t) = \varphi_1(w(t)). \tag{3.3}$$

Substituting (3.3) into the first equation in (2.2), and retaining only the terms of the first and the second order, we obtain

$$\epsilon \frac{d\varphi_0}{dw} \frac{dw}{dt}\bigg|_0 = -A\varphi_0(w) - \epsilon A\varphi_1(W) - f(w). \tag{3.4}$$

Here $\frac{d}{dw}$ denotes the function derivative and $\frac{dw}{dt}|_0$ is the zero-order term in the expansion of $\frac{dw}{dt}$. Unlike w itself its derivative $\frac{dw}{dt}$ is expanded into powers of ϵ.

From (3.4) we see that

$$\varphi_0(w) = -A^{-1} f(w)$$

and expanding the second equation in (2.2) we have

$$\frac{dw}{dt}\bigg|_0 = \varphi_0(w)$$

so that, again from (3.4), we obtain

$$\varphi_1(w) = -A^{-1} \frac{d\varphi_0}{dw} \frac{dw}{dt}\bigg|_0 = -A^{-2} \frac{df}{dw}(w) A^{-1} f(w).$$

Substituting $z^{(1)} = \varphi_0 + \epsilon\varphi_1$ for z and w for x in the first equation in (2.2), we obtain the following system of equations describing the first-order approximation to x

$$\frac{dw}{dt} = -A^{-1}\left(I - \epsilon A^{-1}\frac{df}{dw}(w)A^{-1}\right) f(w). \tag{3.5}$$

To derive the initial condition for (3.5) we have to consider the initial layer functions. We omit the details of the derivation of the relevant equations which can be found in [2], and give here only the equations to be satisfied by $\tilde{x}_0$, $\tilde{x}_1$ and $\tilde{z}_0$. The knowledge of $\tilde{z}_1$ is not relevant for determining the initial condition for (3.5).

We have

$$\frac{d\tilde{x}_0}{d\tau} = 0, \qquad \frac{d\tilde{z}_0}{d\tau} = -A\tilde{z}_0, \qquad \frac{d\tilde{x}_1}{d\tau} = \tilde{z}_0.$$

Since all these are initial layer functions, they have to decay exponentially with τ. Hence, we have to take

$$\tilde{x}_0(\tau) \equiv 0. \tag{3.6}$$

For the remaining two functions we obtain

$$\tilde{x}_0(\tau) = e^{-A\tau}\tilde{x}_0(0) \tag{3.7}$$

and

$$\begin{aligned}\tilde{x}_1(\tau) &= -\int_\tau^\infty \tilde{z}_0(s)\,ds = -\int_\tau^\infty e^{-As}z_0(0)\,ds \\ &= -A^{-1}e^{-A\tau}\tilde{z}_0(0).\end{aligned} \tag{3.8}$$

When integrating, we used the properties of the matrix exponential function.

From (3.6) it is clear that we have to take

$$w(0)|_0 = \alpha.$$

Hence,

$$\bar{z}_0(0) = \varphi_0(\alpha) = -A^{-1}f(\alpha) \quad \text{and} \quad \tilde{z}_0(0) = \beta - \bar{z}_0(0) = \beta + A^{-1}f(\alpha).$$

Thus,

$$\tilde{x}_1(0) = -A^{-1}\tilde{z}_0(0) = -A^{-1}(\beta + A^{-1}f(\alpha)) \quad \text{and} \quad w(0)|_1 = A^{-1}(\beta + A^{-1}f(\alpha)).$$

Finally,

$$\begin{aligned}w(0) &= w(0)|_0 + \epsilon w(0)|_1 \\ &= \alpha + \epsilon A^{-1}(\beta + A^{-1}f(\alpha))\end{aligned} \tag{3.9}$$

is the initial condition for (3.5).

To derive the first-order asymptotic approximation to (2.3) we have to repeat the steps taken when deriving (3.5). However, now the fast variable is x and the slow one is v so that we write

$$x(t) = \bar{x}^{(1)}(t) + \tilde{x}^{(1)}(\tau) + 0(t^2), \qquad v(t) = u(t) + \tilde{v}^{(1)}(\tau) + 0(t^2),$$

where

$$\bar{x}^{(1)}(t) = \bar{x}_0(t) + \epsilon\bar{x}_1(t), \quad \tilde{x}^{(1)}(\tau) = \tilde{x}_0(\tau) + \epsilon\tilde{x}_1(\tau), \quad \tilde{v}^{(1)}(\tau) = \tilde{v}_0(\tau) + \epsilon\tilde{v}_1(\tau).$$

As in (3.3) we define the functions $\psi_0 : \mathbb{R}^n \to \mathbb{R}^n$ and $\psi_1 : \mathbb{R}^n \to \mathbb{R}^n$ and take

$$\bar{x}_0(t) = \psi_0(u(t)), \qquad \bar{x}_1(t) = \psi_1(u(t)),$$

which, substituted into (2.3), yields

$$\epsilon \frac{d\psi_0}{du}\frac{du}{dt}\bigg|_0 = -A\psi_0(u) - \epsilon A\psi_1(u) + u.$$

Hence,

$$\psi_0(u) = A^{-1}u.$$

Since, in the present case,

$$\frac{du}{dt}\bigg|_0 = -f(\psi_0(u)) = -f(A^{-1}u),$$

we have

$$\begin{aligned}\psi_1(u) &= -A^{-1}\frac{d\psi_0}{du}\frac{du}{dt}\bigg|_0 = -A^{-1}A^{-1}\left(-f(A^{-1}u)\right)\\ &= A^{-2}f(A^{-1}u).\end{aligned}$$

Finally, the equation for u is

$$\frac{du}{dt} = -f\left(A^{-1}u + \epsilon A^{-2}fA^{-1}u\right). \tag{3.10}$$

The initial layer equations for $\tilde{v}_0$ and $\tilde{x}_0$ are

$$\frac{d\tilde{v}_0}{d\tau} = 0, \qquad \frac{d\tilde{x}_0}{d\tau} = -A\tilde{x}_0 + \tilde{v}_0. \tag{3.11}$$

As previously, we have to take

$$\tilde{v}_0(\tau) \equiv 0.$$

Hence, in the zero-order approximation,

$$w(0)|_0 = A\alpha.$$

Further, solving the second equation in (3.11), we have

$$\tilde{x}_0(\tau) = e^{-A\tau}\tilde{x}_0(0). \tag{3.12}$$

To find $\tilde{x}_0(0)$, we note that

$$\tilde{x}_0(0) = \psi_0(A\alpha) = A^{-1}A\alpha = \alpha.$$

Hence, we have to take $\tilde{x}_0(0) = 0$ and

$$\tilde{x}_0(\tau) \equiv 0.$$

Then the equation for $\tilde{v}_1$ would be

$$\frac{d\tilde{v}_1}{d\tau} = 0$$

and the solution

$$\tilde{v}_1(\tau) \equiv 0.$$

This shows that, in the first-order approximation, we have to take

$$u(0)|_1 = \beta,$$

so that the initial condition for (3.10) is

$$u(0) = A\alpha + \epsilon\beta. \tag{3.13}$$

Since u is an auxiliary function, the bulk approximation to the solution x of the original, equation (2.3) is obtained from the formula

$$\overline{x}^{(1)}(t) = A^{-1}u(t) + \epsilon A^{-2}f(A^{-1}u). \tag{3.14}$$

If we put $\epsilon = 0$ in (3.5) and (3.9), we obtain the zero-order approximation

$$\frac{dx^{(0)}}{dt} = -A^{-1}f(x^{(0)}), \qquad x^{(0)}(0) = \alpha, \tag{3.15}$$

known in the literature as the steady-state approximation.

The asymptotic convergence of the solutions of the approximate equations derived in this section has been proved by Mika and Palczewski [6] from the assumptions which are quite general and, therefore, somewhat complicated. If we deal, however, with sufficiently smoother functions f such that the solution of the original equation exists over a certain time interval $[0, t_1]$, then the crucial property of (2.1) to be required is that the matrix A has all the eigenvalues with positive real parts, so that the initial layer solutions given by (3.7), (3.8) and (3.12) are exponentially decaying.

From (3.1) we see that the first-order asymptotic solution for x as defined in (2.1) and (2.2) is

$$x^{(1)}(t) = w(t) + \epsilon\tilde{x}_1\left(\frac{t}{\epsilon}\right) \tag{3.16}$$

where w is the solution of (3.5) with the initial condition (3.9) and $\tilde{x}_1$ is given by (3.8). The function $x^{(1)}$ is uniformly convergent to x so that

$$||x(t) - x^{(1)}(t)|| = C_1\epsilon^2, \qquad t \in [0, t_1], \tag{3.17}$$

where C_1 is a constant independent of t and $||\ ||$ is an arbitrary vector norm.

If we are not particularly interested in the behavior of x inside the initial layer, then, after replacing $x^{(1)}$ with w in (3.17), we will have

$$||x(t) - w(t)|| = D_1\epsilon^2, \qquad t \in [t_0, t_1], \tag{3.18}$$

where D_1 is a constant and t_0 is an arbitrary number such that $0 < t_0 < t_1$.

If we are concerned with (2.3) then the first-order asymptotic solution for x is

$$x^{(1)}(t) = \overline{x}^{(1)}(t) + \epsilon\tilde{x}_1\left(\frac{t}{\epsilon}\right), \tag{3.19}$$

where $\overline{x}^{(1)}$ is defined in (3.14) and $\tilde{x}_1$ is the initial-layer solution of the first-order. As previously

$$||x(t) - x^{(1)}(t)|| = C_2\epsilon^2, \qquad t \in [0, t_1] \tag{3.20}$$

and

$$||x(t) - \overline{x}^{(1)}(t)|| = D_2\epsilon^2, \qquad t \in [t_0, t_1]. \tag{3.21}$$

Finally, in the case of (3.15), we have the uniform convergence

$$||x(t) - x^{(0)}(t)|| = C_0\epsilon, \qquad [0, t_1]. \tag{3.22}$$

4. FIRST NUMERICAL EXAMPLE

To check the actual performance of the proposed method, we first consider the system (2.1) in which

$$f_i(x) = x_i^3 + x_{i+1}^3, \qquad i = 1, 2, \ldots, n-1, \qquad f_n(x) = x_n^3 + x_1^3. \tag{4.1}$$

If the matrix A were ill-conditioned, the inverse A^{-1} would have a large norm and that would interfere with the expansion in terms of a small parameter ϵ. To avoid unnecessary complications we will simply take A as the unit matrix.

For the initial condition, we choose

$$\alpha_i = \sin\left(\frac{\pi(i-1)}{n-1}\right), \quad \beta_i = \pi\cos\left(\frac{\pi(i-1)}{n-1}\right), \qquad i = 1, 2, \ldots, n. \tag{4.2}$$

We perform numerical calculations using the stiff method of Gear on a PC/AT. The truncation error has been fixed at 10^{-8}. We take $n = 21$.

Let the direct numerical solutions for x of (2.2) and (2.3) be $\hat{x}^{(1)}$ and $\hat{x}^{(2)}$, respectively. We define the error function

$$\Delta_{\max}(t) = \max_{1\le i\le n}\left|\hat{x}_i^{(1)}(t) - \hat{x}_i^{(2)}(t)\right|, \tag{4.3}$$

which is a good indication of how accurate the numerical integration of (2.2) or (2.3) is. In Table 1 we present typical values of the solution (with six accurate decimal digits) and the corresponding values of $\Delta_{\max}$.

Table 1. The values of x_i and $\Delta_{\max}$ f given by (4.1).

ϵ	t	x_6	x_{11}	$\Delta_{\max}$
0.4	0.15	0.953990	0.953854	8.63E-9
	0.30	1.037480	0.849892	1.59E-8
	1.5	0.298849	0.275277	2.20E-8
	3.0	0.167951	0.190552	1.56E-8
0.2	0.15	0.888640	0.919323	1.66E-8
	0.30	0.859294	0.771597	2.27E-9
	1.5	0.344776	0.324196	2.62E-8
	3.0	0.255003	0.246093	1.83E-8
0.1	0.15	0.800041	0.873543	6.92E-9
	0.30	0.698669	0.706872	6.68E-9
	1.5	0.354928	0.355673	4.61E-8
	3.0	0.263662	0.264436	7.63E-9
0.05	0.15	0.712057	0.830626	2.19E-8
	0.30	0.608068	0.679961	2.04E-8
	1.5	0.349997	0.367831	2.11E-8
	3.0	0.262484	0.271251	1.80E-8
0.0025	0.15	0.655158	0.806524	5.72E-9
	0.30	0.571358	0.675456	1.63E-8
	1.5	0.345285	0.372942	1.98E-8
	3.0	0.260660	0.274192	4.20E-8

To compare the approximate solutions with the exact one we define two error functions

$$\Delta_{\max}^{(k)}(t) = \max_{1\le i\le n}\left|x_i(t) - y_i^{(k)}(t)\right|, \quad \Delta_{av}^{(i)}(t) = \frac{1}{n}\sum_{i=1}^{n}\left|x_i(t) - y_i^{(k)}(t)\right|, \qquad k = 0, 1, 2, \tag{4.4}$$

where x stands for $\hat{x}^{(1)}$ or $\hat{x}^{(2)}$ which appear in (4.3). For simplicity, we use the notation $y^{(0)}(t) = x^{(0)}(t)$ as defined in (3.15), $y^{(1)}(t) = w(t)$ as defined in (3.16) and $y^{(2)}(t) = \overline{x}^{(1)}$ as defined in (3.14). All these functions represent the bulk component of the asymptotic solution to x.

The values of the error functions for various ϵ and t are given in Table 2. From Table 2 we see that the proposed method might be of interest for the values of ϵ which are neither too large nor too small. In the first case, it is perfectly reasonable to integrate directly the original system of equations (2.2) or (2.3) and, in the second case, it is advisable to use the zero-order approximation.

Table 2. Errors of approximate solutions f given by (4.1).

t	ϵ	Δ^0_{max}	Δ^1_{max}	Δ^2_{max}	Δ^0_{av}	Δ^1_{av}	Δ^2_{av}
0.15	0.4	3.93E-1		6.39E-1	2.77E-1		3.48E-1
	0.2	3.92E-1		2.98E-1	2.28E-1		1.93E-1
	0.1	2.44E-1	2.63E-1	1.55E-1	1.63E-1	1.00E-1	8.17E-2
	0.05	1.49E-1	6.85E-2	7.38E-2	9.21E-2	2.49E-2	3.29E-2
	0.025	7.83E-2	3.37E-2	3.20E-2	4.50E-2	1.22E-2	1.36E-2
0.3	0.4	6.63E-1		5.53E-1	4.24E-1		3.49E-1
	0.2	4.88E-1		2.92E-1	2.90E-1		1.52E-1
	0.1	2.98E-1	2.23E-1	1.18E-1	1.60E-1	1.02E-1	5.36E-2
	0.05	1.57E-1	8.69E-2	3.90E-2	7.61E-2	3.32E-2	1.89E-2
	0.025	7.85E-2	3.96E-2	1.58E-2	3.65E-2	1.53E-2	8.13E-3
1.5	0.4	8.24E-1		5.42E-1	2.93E-1		1.04E-1
	0.02	5.33E-1		9.76E-2	1.71E-1		2.76E-2
	0.1	3.01E-1	1.35E-1	1.55E-2	8.97E-2	5.34E-2	6.88E-3
	0.05	1.56E-1	3.90E-2	5.77E-3	4.52E-2	1.85E-2	2.97E-3
	0.025	7.84E-2	1.74E-2	2.48E-3	2.25E-2	8.41E-3	1.55E-3
3.0	0.4	6.12E-1		1.37E-1	2.11E-1		5.71E-2
	0.2	4.47E-1		4.26E-2	1.17E-1		1.33E-2
	0.1	2.78E-1	6.02E-2	6.84E-3	6.65E-2	2.69E-2	3.31E-3
	0.05	1.53E-1	1.96E-2	2.60E-3	3.47E-2	1.02E-2	1.33E-3
	0.025	7.80E-2	8.95E-3	1.07E-3	1.75E-2	4.73E-3	6.76E-4

The proposed method is indeed of the second order, although the errors do not behave like ϵ^2 for small times due to the existence of the initial-layer solution and for very small values of ϵ due to the truncation errors.

It is difficult to decide which of the systems, (2.2) or (2.3), is more adequate for the application of the asymptotic expansion technique. The second approach is slightly simpler algorithmically, since in that case there is no need to calculate the Jacobi matrix $\frac{df}{dx}$. On the other hand, the original solution x is expressed in terms of the fast variable and this might be the source of additional truncation errors and of the increase in the computing time.

Table 2 does not contain the results for (3.5) and $\epsilon = 0.4$ and $\epsilon = 0.2$. In those cases, the error estimate inside the solver was too big and the time step became increasingly small. That shows again that for large ϵ one has to be careful with using the present method.

The application of the proposed method for small values of ϵ leads to the reduction of the numerical effort. For example, for $\epsilon = 0.1$ and $t = 3$, the direct solution of (2.2) took $246\,s$, whereas the solution of (3.5) was obtained in $92\,s$. The reduction of computing time might be crucial when one attempts to solve large systems of differential equations on a PC.

5. SECOND NUMERICAL EXAMPLE

The seminal system of equations (2.1) considered in this paper is closely related to the singularly perturbed telegraph equation which can be written in the following form

$$\epsilon \frac{\partial^2 q}{\partial t^2} + A\frac{\partial q}{\partial t} = \mu \frac{\partial^2 q}{\partial \rho^2} - \gamma(t) q(\rho, t), \tag{5.1}$$

where the unknown function q depends on a spatial variable ρ and time t, A is a constant matrix, μ is an arbitrary constant and γ an arbitrary function of t.

For simplicity, we assume that the spatial variable ρ changes from 0 to 1 and divide the interval $[0, 1]$ into $n - 1$ equal subintervals. If we define

$$q\left(\frac{i-1}{n-1}, t\right) = x_i(t), \qquad i = 1, 2, \ldots, n, \tag{5.2}$$

Table 3. The values of x_i and $\Delta_{\max} f$ given by (5.3).

ϵ	t	x_6	x_{11}	$\Delta_{\max}$
0.4	0.15	0.964453	0.974710	1.05E-7
	0.30	1.095972	0.911454	2.92E-8
	1.5	0.695759	0.321959	6.10E-8
	3.0	0.916587	0.379776	4.28E-8
0.2	0.15	0.905342	0.955061	9.62E-8
	0.30	0.927509	0.858385	2.08E-8
	1.5	0.406668	0.314338	2.09E-8
	3.0	0.688973	0.530286	9.11E-9
0.1	0.15	0.822151	0.927523	2.09E-8
	0.30	0.753185	0.804550	4.52E-8
	1.5	0.318225	0.332239	5.70E-8
	3.0	0.624641	0.649817	6.57E-9
0.05	0.15	0.734047	0.898616	8.25E-9
	0.30	0.633777	0.769167	1.55E-8
	1.5	0.283848	0.344171	7.83E-9
	3.0	0.600290	0.724637	4.40E-10
0.025	0.15	0.670955	0.878271	7.62E-8
	0.30	0.575341	0.752929	2.23E-8
	1.5	0.267918	0.350488	2.50E-10
	3.0	0.589275	0.767042	1.54E-8

Table 4. Errors of approximate solutions f given by (5.3).

t	ϵ	$\Delta^0_{\max}$	$\Delta^1_{\max}$	$\Delta^2_{\max}$	Δ^0_{av}	Δ^1_{av}	Δ^2_{av}
0.15	0.4	4.14E-1	6.61E-1	6.82E-1	2.65E-1	4.21E-1	4.22E-1
	0.2	3.45E-1	2.18E-1	2.37E-1	2.21E-1	1.33E-1	1.51E-1
	0.1	2.48E-1	4.18E-2	8.77E-2	1.60E-1	2.00E-2	5.47E-2
	0.05	1.46E-1	1.08E-2	4.72E-2	9.47E-2	4.29E-3	2.94E-2
	0.025	7.32E-2	5.10E-3	2.35E-2	4.78E-2	2.42E-3	1.45E-2
0.3	0.4	6.64E-1	2.49E-1	3.16E-1	4.32E-1	1.20E-1	2.00E-1
	0.2	4.67E-1	4.94E-2	1.63E-1	3.06E-1	1.82E-2	1.01E-1
	0.1	2.68E-1	4.76E-2	9.40E-2	1.75E-1	4.28E-2	5.89E-2
	0.05	1.30E-1	1.64E-2	4.29E-2	8.53E-2	8.68E-3	2.66E-2
	0.025	6.18E-2	4.39E-3	1.95E-2	4.07E-2	2.05E-3	1.19E-2
1.5	0.4	6.84E-1	3.55E-1	3.63E-1	4.34E-1	2.24E-1	2.27E-1
	0.2	2.65E-1	8.74E-2	7.22E-2	1.72E-1	5.59E-2	4.69E-2
	0.1	1.19E-1	3.32E-2	1.73E-2	7.86E-2	2.09E-2	9.81E-3
	0.05	5.82E-2	1.61E-2	5.41E-3	3.83E-2	1.03E-2	2.44E-3
	0.025	2.89E-2	8.13E-3	2.02E-3	1.89E-2	5.27E-3	7.77E-4
3.0	0.4	1.02E-0	4.33E-1	3.59E-1	6.58E-1	2.79E-1	2.30E-1
	0.2	5.30E-1	2.13E-1	9.91E-2	3.40E-1	1.35E-1	6.34E-2
	0.1	2.85E-1	1.27E-1	5.56E-2	1.82E-1	8.42E-2	3.76E-2
	0.05	1.51E-1	7.12E-2	3.25E-2	9.64E-2	4.75E-2	2.26E-2
	0.025	7.83E-2	3.79E-2	1.79E-2	4.99E-2	2.54E-2	1.24E-2

then (5.1) can be written in the form (2.1) with

$$f(x) = -\mu Dx + \gamma(t)x. \tag{5.3}$$

The matrix D represents the central difference approximation to the second derivative and has the form

$$(Dx)_i = (n-1)^2(x_{i-1} - 2x_i + x_{i+1}), \qquad i = 2, 3, \ldots, n-1. \tag{5.4}$$

The form of $(Dx)_1$ and $(Dx)_n$ depends on the boundary condition which is to be satisfied by q in (5.1).

Here we take the zero-current condition

$$\frac{\partial q}{\partial \rho}(0,t) = \frac{\partial q}{\partial p}(1,t) = 0,$$

which gives

$$(Dx)_1 = (n-1)^2(2x_2 - 2x_1), \qquad (Dx)_n = (n-1)(2x_{n-1} - 2x_n).$$

We take the initial condition for (5.1) as

$$q(\rho,0) = \sin \pi\rho, \qquad \frac{\partial q}{\partial t}(\rho,0) = \pi \cos \pi\rho,$$

which, upon discretisation, gives for the same initial condition as in (4.2).

As in the previous case, we take A as the unit matrix. Since the matrix D is multiplied by a large number $(n-1)^2$, we have to take μ relatively small to avoid introducing large coefficients into the system and upsetting the asymptotic expansion procedure. Thus, our approach is applicable to those telegraph equations for which the diffusion is relatively small. In fact, we take $\mu = 0.02$.

The results of calculations are given in Tables 3 and 4. The notation is exactly the same as in the previous section and the conclusions very similar.

References

1. A.B. Vasil'eva and V.F. Butuzov, *Asymptotic Expansions of Solutions of Singularly Perturbed Equations*, (in Russian), Nauka, Moscow, (1973).
2. R.E. O'Malley, Jr., *Introduction to Singular Perturbations*, Academic Press, New York, (1974).
3. D.R. Smith, *Singular Perturbation Theory*, Cambridge University Press, Cambridge, (1985).
4. J.R. Mika, Application of the Chapman-Enskog method to an asymptotic analysis of stiff systems of ordinary differential equations, In *Proceedings of the Conference on Computational Ordinary Differential Equations*, Oxford University Press, London, Oxford, (July 3–7, 1989) (to appear).
5. J.R. Mika and A. Palczewski, Application of the asymptotic expansion method for singularly perturbed equations of the resonance type in the kinetic theory, *Archiwa Mechaniki* **35** (3), 395–408 (1983).
6. J.R. Mika and A. Palczewski, Asymptotic analysis of singularly perturbed systems of ordinary differential equations, *Computers and Mathematics with Applications* **21** (10), 13–32 (1991).
7. R.C. Aiken, Editor, *Stiff Computation*, Oxford University Press, New York, (1985).